Standard cosmological model with non vanishing Weyl tensor

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We have solved Einstein's equations of general relativity for a homogeneous and isotropic metric with constant spatial curvature and found a non vanishing Weyl tensor in the presence of an anisotropic pressure component of the energy-momentum tensor. The time evolution of the spacetime is guided by the usual Friedman equations and the properties of the spatial components comprise a separated system of equations that can be independently solved. The physical features of this solution are elucidated by using the Quasi-Maxwellian equations of general relativity which directly connect the anisotropic pressure to the electric part of the Weyl tensor for the cosmological fluid.

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I. INTRODUCTION

In the literature we can find at least two formulations of the Einstein gravitational theory: one of them is the general relativity (GR) [1] and the other one is the Quasi-Maxwellian (QM) equations [2]. According to Lichnerowicz's theorem [3], these formulations are equivalent if appropriate initial conditions are used. It is clear that the theorem constrains the solutions of the Quasi-Mawellian equations in order that the two formulations surely give the same results. The main point of this restriction appears when we want to determine the initial data in terms of observable quantities aiming to guarantee that the solution we shall obtain has something to do with the empirical problem we started with. It is well known that the observables in general relativity can only be determined by the geodesic deviation and that they are represented by the curvature tensor. This tensor has 20 independent components that can be expressed in terms of the Ricci tensor, the scalar curvature and the Weyl tensor. In the usual formulation of Einstein's equations only the Ricci tensor and the scalar curvature are involved. In this case, the Weyl tensor remains completely undetermined—since the initial data are g_{ij} and $g_{ij,0}$ (up to a diffeomorphism), which cannot be associated to the empirical data corresponding to the Weyl tensor. In other words, it is not possible to look for a solution of general relativity that corresponds to a determined Weyl tensor specified by a set of empirical data. This information cannot be expressed in terms of the initial values of the variables of the Einstein equations.

All these difficulties disappear in the Quasi-Maxwellian formulation of gravity. In this approach, the variables of the theory represent directly the empirical data and the Einstein equations are used to relate the Weyl tensor to the energy-momentum tensor through differential equations involving both. In particular, in this paper we present how this relation takes place when we analyze the standard cosmological model (SCM) closely.

Moreover, the SCM experiences difficulties in which exotic components of matter and energy are introduced as an attempt to explain, for instance, the apparent accelerated expansion of the universe (dark energy), the galaxy rotation curves (dark matter) or the initial singularity in the past. Some authors claim that the main problem concerns the huge simplification of the geometry adopted and hence they suggest modifications in the spacetime symmetries as presented by the inhomogeneous models [4], in particular indicating different average processes [5], or modifications in the coupling between matter and geometry as shown by the bouncing cosmologies [6], or even nonconventional proposals as [7], which change completely our understanding upon the spacetime itself.

Notwithstanding, our proposal is simpler. Instead of assuming modifications of geometry, we show that Robertson-Walker (RW) metrics admit a more general solution when we introduce an anisotropic pressure term $\pi_{\mu\nu}$ in the Einstein equations. Such term has not been considered in the establishment of the cosmological model by two simple reasonings: first, it apparently breaks the homogeneity and isotropy of the spacetime; second, in the case of free-shear geometries (e.g. the Friedman metric) there is no other traceless symmetric tensor phenomenologically linked to $\pi_{\mu\nu}$. Both arguments are not true and we shall see that this term exemplifies exactly what we previously pointed out about the problem of initial conditions in GR.

Summarizing, in the present paper we show that if one considers a RW metric and an energy-momentum tensor of a perfect fluid with an anisotropic pressure, the QM-equations allow a more general solution of the Einstein equations which possesses, on one hand, a time evolution given by the usual Friedmann equations and, on the other hand, a nonzero Weyl tensor given in terms of the anisotropic pressure.

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II. FRIEDMAN SOLUTION REVISITED

The observational data indicates that the Friedman geometry is more convenient to describe the homogeneity and isotropy detected at large scales of the Universe. These symmetries suggest that the only possible fluid satisfying these properties is a perfect fluid with ρ (energy density) and p (isotropic pressure). However, as we mentioned before, this is not a paradigm. In parts this is the scope of our paper in the sense that we will not consider only a perfect fluid as source for the gravitational field.

To make a self-consistent exposition of our results, we present here a brief derivation of the Friedman model. Thus, let us start considering the infinitesimal line element given by

$$ds^2 = dt^2 - a^2(t)[d\chi^2 + \sigma^2(\chi)d\Omega^2]. \tag{1}$$

where t represents the cosmic time and $a(t)$ is the scale factor. A straightforward calculation gives the following scalar curvature

$$R = 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} - \frac{2}{a^2}\left(2\frac{\sigma''}{\sigma} + \frac{\sigma'^2}{\sigma^2} - \frac{1}{\sigma^2}\right). \tag{2}$$

where dot means time derivative and prime means derivative w.r.t. χ. The spatial curvature ${}^{(3)}R$ of the hypersurface defined by $t \equiv const.$ is

$${}^{(3)}R = -2\left(2\frac{\sigma''}{\sigma} + \frac{\sigma'^2}{\sigma^2} - \frac{1}{\sigma^2}\right). \tag{3}$$

Assuming that ${}^{(3)}R$ has the same value everywhere, we set

$${}^{(3)}R = 6\epsilon,$$

where ϵ is a constant. Therefore, the scalar curvature becomes

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\epsilon}{a^2}\right). \tag{4}$$

This equation shows that the scalar curvature of the spacetime depends only on time.

The energy-momentum distribution is described by a perfect fluid with energy density ρ, isotropic pressure p and comoving 4-velocity $V^\mu = \delta^\mu_0$, namely

$$T_{\mu\nu} = (\rho + p)V_\mu V_\nu - p g_{\mu\nu}.$$

In general, one assumes the existence of an equation of state such that $p = \lambda\,\rho$, where λ is a constant. In this way, we can calculate the nonnull components of Einstein's equations $G^\mu{}_\nu = -T^\mu{}_\nu$ (with Einstein's constant equal to 1), which are explicitly given by

$$3\frac{\dot{a}^2}{a^2} + 3\frac{\epsilon}{a^2} = \rho, \tag{5a}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{3\epsilon}{a^2} + \frac{2}{a^2}\frac{\sigma''}{\sigma} = -p, \tag{5b}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{1}{a^2}\frac{\sigma''}{\sigma} = -p. \tag{5c}$$

Subtracting (5b) from (5c) yields

$$\frac{\sigma''}{\sigma} + \epsilon = 0, \tag{6}$$

Alternatively, we can combine Eqs. (6) and (3) to obtain the equation

$$\frac{\sigma''}{\sigma} - \frac{\sigma'^2}{\sigma^2} + \frac{1}{\sigma^2} = 0. \tag{7}$$

One can see that each value for the spatial curvature corresponds to a single curve in the space of solutions. It obviously happens because Eqs. (6) and (7) must be satisfied at the same time. Remark that solutions with the same sign for ϵ are topologically equivalent. It means that $sign(\epsilon)$ is enough to characterize all the solutions of Eq. (6). Therefore, these equations have only three relevant solutions, i.e.

$$\begin{cases} \epsilon = 0 & \Longrightarrow\ \sigma = \chi, \\ \epsilon = 1 & \Longrightarrow\ \sigma = \sin\chi, \\ \epsilon = -1 & \Longrightarrow\ \sigma = \sinh\chi. \end{cases} \tag{8}$$

We can make a coordinate transformation given by $r = \sigma(\chi)$ to explicitly exhibit the metric in spherically symmetric coordinates. The line element thus becomes

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1 - \epsilon r^2} + r^2 d\Omega^2\right). \tag{9}$$

Note that this metric is conformally equivalent to Minkowski, de Sitter and anti-de Sitter for ϵ equal to 0, -1 and 1, respectively. Using these coordinates, the time evolution of the scale factor is given by

$$H^2 = \frac{\rho}{3} - \frac{\epsilon}{a^2}, \tag{10a}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(1 + 3\lambda)\rho, \tag{10b}$$

where $H \equiv \dot{a}/a$ is called the Hubble parameter. The system of equations (10) corresponds to the well-known

Friedman equations. A standard analysis of this model can be found in [8] and references therein.

In terms of the Quasi-Maxwellian formalism, the Friedman equations are equivalent to

$$\dot{\theta} + \frac{\theta^2}{3} = -\frac{1}{2}(1+3\lambda)\rho, \tag{11a}$$

$$\dot{\rho} + (1+\lambda)\rho\,\theta = 0, \tag{11b}$$

which are the Raychaudhuri equation and the continuity equation, respectively.

III. FRIEDMAN MODEL IN THE PRESENCE OF AN ANISOTROPIC PRESSURE

In this section we basically reproduce last section adding to the Einstein equations an extra term on the right hand side corresponding to the presence of the anisotropic pressure $\pi_{\mu\nu}$. In the formalism of fluid mechanics [9], this term describes all processes involving viscosity and, consequently, energy dissipation. Notwithstanding, with the advent of the SCM in the eighties, we usually do not find sources for the gravitational field including such term anymore. At most, we can see works dealing with corrections to the isotropic pressure producing a Stokesian type fluid [10, 11]. Concerning only the thermodynamics, some references analyze phase transitions produced by the gravitational field in the presence of the anisotropic pressure [12, 13]. However, none of them take into account the anisotropic pressure on the right hand side of the Einstein equations.

To do so, we propose that the most general source for the Friedman geometry is represented by[1]

$$T_{\mu\nu} = (\rho + p)V_\mu V_\nu - p\,g_{\mu\nu} + \pi_{\mu\nu}.$$

Starting from the line element given by Eq. (1), the non-trivial components of Einstein's equations are

$$3\frac{\dot{a}^2}{a^2} + 3\frac{\epsilon}{a^2} = \rho, \tag{12a}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{3\epsilon}{a^2} + \frac{2}{a^2}\frac{\sigma''}{\sigma} = -p + \pi^1{}_1, \tag{12b}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{1}{a^2}\frac{\sigma''}{\sigma} = -p + \pi^2{}_2. \tag{12c}$$

Subtracting (12b) from (12c) and using the traceless condition $\pi^\mu{}_\mu = 0$, we obtain

$$\frac{\sigma''}{\sigma} + \epsilon = \frac{1}{2}f(\sigma), \tag{13}$$

where we assumed that

$$\pi^1{}_1 \equiv \frac{f(\sigma)}{a^2}.$$

The factor $1/a^2$ in the expression of π^1_1 is introduced for consistency reasons. Alternatively, we can combine Eqs. (13) and (3) to obtain

$$\frac{\sigma''}{\sigma} - \frac{\sigma'^2}{\sigma^2} + \frac{1}{\sigma^2} = \frac{3}{2}f. \tag{14}$$

However, the solutions of Eqs. (3) and (14) are the same only if we impose

$$f = \frac{2k}{\sigma^3},$$

where k is an integration constant. This condition emerges from the compatibility relation of the first integrals of both equations

$$\frac{\sigma'^2}{\sigma^2} - \frac{1}{\sigma^2} + \epsilon + \frac{2k}{\sigma^3} = 0. \tag{15}$$

Making the coordinate transformation given by $r = \sigma(\chi)$, the line element (1) becomes

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1 - \epsilon r^2 - \frac{2k}{r}} + r^2 d\Omega^2\right). \tag{16}$$

One can see that the time evolution of the RW metric is still given by the usual Friedman equations (10a) and (10b). When $k = 0$ the Friedman model is completely recovered. Note that this solution is not conformally equivalent to Schwarzschild-de Sitter metric. Indeed, it is a cosmological solution with new features.

IV. QUASI-MAXWELLIAN EQUATIONS AND RW METRICS

In this section, we use the Quasi-Maxwellian equations of gravity to derive the solution expressed by Eq. (16). It is convenient to assume an observer field as $V^\mu = \delta^\mu_0$. The energy momentum distribution is given a perfect fluid with anisotropic pressure $\pi_{\mu\nu}$. Using the metric represented by Eq. (1), these assumptions lead to the following non vanishing Quasi-Maxwellian equations (see details in Appendix):

[1] We cannot add the heat flux q^μ because it breaks the isotropy of the RW space-time.

$$\dot{\theta} + \frac{\theta^2}{3} = -\frac{1}{2}(\rho + 3p), \tag{17a}$$

$$\dot{\rho} + (\rho + p)\,\theta = 0, \tag{17b}$$

$$E_{\mu\nu} = -\frac{1}{2}\pi_{\mu\nu}, \tag{17c}$$

$$E^{\alpha}{}_{\mu;\alpha} = 0. \tag{17d}$$

As we said before, Eqs. (17a) and (17b) correspond to the Friedman equations. Eq. (17c) yields the electric part of the Weyl tensor in terms of the anisotropic pressure. The equation (17d) represents the compatibility condition of the Einstein equations with the (constant) spatial curvature given by Eq. (15). Remark that this equation does not imply that the Weyl tensor is identically zero.

From a straightforward calculation, the electric part of Weyl tensor is read as

$$[E^i{}_j] = E(t,\chi)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \tag{18}$$

where

$$E(t,\chi) = -\frac{1}{3a^2}\left(\frac{\sigma''}{\sigma} - \frac{\sigma'^2}{\sigma^2} + \frac{1}{\sigma^2}\right). \tag{19}$$

Substituting this expression into Eq. (17d), we have to solve an extra constraint, which is given by

$$-\frac{1}{3a^2}\left(\frac{\sigma''}{\sigma} - \frac{\sigma'^2}{\sigma^2} + \frac{1}{\sigma^2}\right) = \frac{h(t)}{\sigma^3}, \tag{20}$$

where $h(t)$ is an arbitrary function. Multiplying both sides of Eq. (20) by $a^2\sigma^3$, the following relation appears

$$\sigma^2\sigma'' - \sigma\sigma'^2 + \sigma = -3a^2(t)h(t). \tag{21}$$

The equation for the time coordinate is trivially satisfied by fixing $h = -k/a^2$, where k is a constant. To solve the equation for the spatial coordinate χ, we use Eq. (3), the spatial curvature in terms of σ, and rewrite Eq. (21) as follows

$$\frac{\sigma''}{\sigma} + \epsilon - \frac{k}{\sigma^3} = 0, \tag{22}$$

Multiplying both sides of this equation by $2\sigma\sigma'$, we get a first integral of this form:

$$\sigma'^2 + \epsilon\sigma^2 + \frac{2k}{\sigma} = C_1, \tag{23}$$

The first integral coming from the (constant) spatial curvature equation is expressed by

$$\sigma'^2 + \epsilon\sigma^2 - 1 - \frac{C_2}{\sigma} = 0, \tag{24}$$

The common solutions are obtained by setting $C_1 = 1$ and $C_2 = -2k$.

Finally, we make the coordinate transformation $r = \sigma(\chi)$ to rewrite the line element (1) as given by Eq. (16). Note that this solution has a non vanishing electric part of the Weyl tensor given by

$$[E^i{}_j] = \frac{k}{a^2r^3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \tag{25}$$

and the spatial curvature ${}^{(3)}R$ remains constant. Remark that Eq. (25) is very similar to the Newtonian tidal forces times a time dependent function $1/a^2$. The consequences of this fact will be discussed in next sections.

V. TRAJECTORIES OF TEST PARTICLES

Instead of integrating completely the geodesic equations of the metric (16), which seems unnecessary at this moment, a qualitative analysis of the particle trajectories in this geometry can be done. In fact, we only analyze the geodesic motion of test particles (timelike and nulllike) moving along the equatorial plane ($\theta = \pi/2$ and $\dot{\theta} = 0$). This simplification allows to study the behavior of the effective potential to which these particles are subjected and comparison to other cases can be easily done.

In this way, the simplified geodesic equations reduce to

$$t'' + \frac{a\dot{a}}{A}r'^2 + a\dot{a}r^2\phi'^2 = 0, \tag{26}$$

$$t'^2 - \frac{a^2}{A}r'^2 - a^2r^2\phi'^2 = b, \tag{27}$$

$$\phi'' + 2\frac{\dot{a}}{a}t'\phi' + 2\frac{r'}{r}\phi' = 0, \tag{28}$$

where we denoted $X' \equiv dX/d\tau$, τ is the affine parameter along the curve (which is the proper time for the timelike geodesics) and b is equal to 1 for timelike geodesics or 0 for nulllike ones. We also denoted

$$A(r) = 1 - \epsilon\, r^2 - \frac{2k}{r}.$$

Remark that we do not fix the value of the spatial curvature in the analysis below: ϵ remains an arbitrary constant.

First, we solve Eq. (28) and see that the angular momentum is a conserved quantity

$$\phi' = \frac{l}{a^2 r^2}, \tag{29}$$

where l is an integration constant. Then, substituting Eqs. (29) and (27) into Eq. (26) we get

$$t'' + \frac{\dot{a}}{a}(t'^2 - b) = 0, \tag{30}$$

which can be integrated, yielding

$$(t'^2 - b) = -\frac{E}{a^2}, \tag{31}$$

where E is another integration constant, in general associated to the total energy of the test particle[2]. Substituting this equation in Eq. (27) results

$$a^4 r'^2 = -\left(E + \frac{l^2}{r^2}\right) A. \tag{32}$$

This equation can be seen as the energy conservation equation of a particle moving in a one-dimensional effective potential. Note that this equation has a constant term E (like a mechanical energy), a kinetic-like term $a^4 r'^2$ and the remaining ones correspond to the effective potential that we denote by $V(r)$.

To compare Eq. (32) with the effective potential obtained from the Schwarzschild metric [14], we suppose the spatial curvature equal to zero ($\epsilon = 0$). Therefore, Eq. (32) becomes

$$a^4 r'^2 = -E + \frac{2kE}{r} - \frac{l^2}{r^2} + \frac{2kl^2}{r^3}. \tag{33}$$

The effective potential $V(r)$ can be written down as

$$V(r) = -\frac{2kE}{r} + \frac{l^2}{r^2} - \frac{2kl^2}{r^3}.$$

Note that this potential presents the same terms as the Schwarzschild geometry in the case of a single particle moving along a geodesic in this metric. In particular, both potentials have the same qualitative behavior if we choose $k > 0$ and $E > 0$ and, therefore, they produce the same particle trajectories.

For the sake of completeness and comparison, we start a more detailed, but still qualitative, analysis of the Kepler problem in this solution rewriting Eq. (32) in terms of the variable

$$u = \frac{1}{r},$$

where we seek for the planetary orbits $u = u(\phi)$. However, the time-dependence of Eq. (32) makes the problem more complicated. Therefore, we consider an interval of cosmological time (t_0, t_1) in which the scale factor remains almost constant, i.e., $a(t) \equiv a_0$ for $t \in (t_0, t_1)$. In other words, we assume that the cosmological time changes very slowly in comparison to the period of the planetary orbits. We thus obtain

$$l^2 \left(\frac{du}{d\phi}\right)^2 = -(E + l^2 u^2)(1 - 2ku). \tag{34}$$

Differentiating this equation with respect to ϕ, yields

$$\frac{d^2 u}{d\phi^2} + u = 3ku^2 + \frac{Ek}{l^2}. \tag{35}$$

This is exactly the differential equation given by the Schwarzschild metric in response to the Kepler problem if we identify k to the effective mass of the gravitational source and E is identified to the total mass of the test particle. In particular, considering $E = 1$ we can correctly reproduce the perihelion shift and for $E = 0$ we obtain the light deflection predicted by general relativity via Schwarzschild metric. Note that b and E practically coincide in this regime. In this way, we suggest that dark matter effects can be interpreted as a convenient choice of the Weyl tensor as shown by Eq. (35) or even that dark matter is only a problem related to the bad choice of initial conditions in GR.

VI. GEODESIC DEVIATION OF THE COSMOLOGICAL FLUID

Among its attributes, the Quasi-Maxwellian representation of gravity has the quality to put together the formalism of the electromagnetic interactions with a formal approach of GR. Nonetheless some fundamental distinctions must be stressed. The empirical determination of an electromagnetic field is made through the Lorentz force and a test particle in order to identify the presence of an electromagnetic field. The electromagnetic tensor, obtained from the integration of Maxwell equations, does not distinguish the contribution of the local charges and currents distribution from boundary conditions. In GR the empirical identification of a gravitational field cannot be done using a single test particle since the Christoffel symbols can be set equal to zero by coordinate transformations. In order to empirically determine the properties of the gravitational field it is necessary to look for the geodetic deviation expressed in terms of the curvature tensor. This tensor explicitly separate the contribution coming form the local distribution of the energy-momentum tensor, algebraically associated to the traces of the Riemann tensor, from the global contribution of boundary conditions represented here by the

[2] Opposite to the Schwarzschild case, the first integral of the time component of the geodesic equation is not necessarily positive definite.

Weyl tensor. Indeed, the measurements of the gravitational field effects can be only done through the geodesic deviation equation which determines the rate of the relative acceleration between infinitesimally near geodesics. This equation is then given by

$$\frac{d^2 z^\alpha}{ds^2} = -R^\alpha{}_{\beta\mu\nu} V^\beta z^\mu V^\nu. \qquad (36)$$

where z^α is the deviation vector and V^μ is the vector field tangent to the geodesic congruence.

The distortion produced by the Weyl tensor upon a given congruence of curves can only be detected by this equation substituting the Riemann tensor by its decomposition into irreducible parts: the Ricci tensor, scalar curvature and the Weyl tensor.

Since we are dealing with the cosmological fluid we set $V^\mu = \delta^\mu_0$, evaluate the right hand side of Eq. (36) and get

$$\frac{d^2 z^\alpha}{ds^2} = \left(E^\alpha{}_\mu + \frac{1}{2}\pi^\alpha{}_\mu \right) z^\mu + \frac{1}{6}(\rho + 3p) h^\alpha{}_\mu z^\mu.$$

Now comes a remarkable result: according to Eq. (17c) the term inside big brackets is identically zero. Therefore, the cosmological fluid does not measure any distortion caused by the presence of the anisotropic pressure. In other words, the distortion caused by the anisotropic pressure and the electric part of the Weyl tensor are compensated in such a way that the cosmological observer does not attribute any eventual modification of the space-time to these quantities, enabling one to set $E_{\mu\nu}$ and $\pi_{\mu\nu}$ equal to zero by hand. However, this is not allowed if we want to understand correctly the gravitational field effects in the Universe using the empirical data as initial conditions.

VII. CONCLUDING REMARKS

We could see that general relativity does not contain in its dynamics all the information necessary to determine the curvature tensor from empirical data. It means that the initial condition problem in GR (also called Cauchy problem) should be revisited, since a more realistic description of the Universe must be privileged (an open system rather than a close totality).

In particular, in this paper, we showed how the standard cosmological model sets the Weyl tensor equal to zero *ab initio* and that it is not consequence of the Friedmann equations. Therefore, we have developed a cosmological model with constant spatial curvature and nonzero Weyl tensor without wasting homogeneity and isotropy.

VIII. APPENDIX: QUASI-MAXWELLIAN EQUATIONS

Bianchi's identities together with the Einstein equations of gravity yield the Quasi-Maxwellian equations if we rewrite the Riemann tensor in terms of its traces and the Weyl tensor. Assuming Einstein's constant equal to 1, these equations are

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = -\frac{1}{2} T^{\mu[\alpha;\beta]} + \frac{1}{6} g^{\mu[\alpha} T^{,\beta]}. \qquad (37)$$

The Weyl tensor is given by

$$W_{\alpha\beta\mu\nu} \doteq R_{\alpha\beta\mu\nu} - M_{\alpha\beta\mu\nu} + \frac{1}{6} R g_{\alpha\beta\mu\nu},$$

where the auxiliary tensors are

$$2M_{\alpha\beta\mu\nu} \doteq R_{\alpha\mu} g_{\beta\nu} + R_{\beta\nu} g_{\alpha\mu} - R_{\alpha\nu} g_{\beta\mu} - R_{\beta\mu} g_{\alpha\nu}$$

and

$$g_{\alpha\beta\mu\nu} \doteq g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}.$$

The reason of this nomenclature is due to several analogies between the Quasi-Maxwellian and the Maxwell equations. However, this similarity does not go far because QM-equations are in fact highly non-linear in comparison to Maxwell's theory, leading to situations that never happen in the last case.

The projection of the QM-equations with respect to the vector field V^α and its orthogonal hypersurface allows us to do calculations with. At this point, it is very useful to replace the Weyl tensor by its electric $E_{\alpha\beta}$ and magnetic $H_{\alpha\beta}$ parts, respectively:

$$\begin{aligned} E_{\alpha\beta} &\doteq -W_{\alpha\mu\beta\nu} V^\mu V^\nu, \\ H_{\alpha\beta} &\doteq -{}^*W_{\alpha\mu\beta\nu} V^\mu V^\nu, \end{aligned}$$

where ${}^*W_{\alpha\mu\beta\nu}$ is the dual of Weyl tensor constructed using the skew-symmetric Levi-Civita tensor.

The covariant derivative of V^μ can be decomposed into its irreducible parts:

$$V_{\mu;\nu} = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu} + a_\mu V_\nu,$$

where $\theta \equiv V^\mu{}_{;\mu}$ is the expansion coefficient, $a^\mu \equiv V^\mu{}_{;\nu} V^\nu$ is the acceleration,

$$\sigma_{\mu\nu} \equiv \frac{1}{2} h_\mu{}^\alpha h_\nu{}^\beta V_{(\alpha;\beta)} - \frac{\theta}{3} h_{\mu\nu}$$

is the shear tensor and

$$\omega_{\mu\nu} \equiv \frac{1}{2} h_\mu{}^\alpha h_\nu{}^\beta V_{[\alpha;\beta]}$$

is the vorticity. Therefore, four independent projections lead to the following linearly independent equations

$$h^{\epsilon\alpha}h^{\lambda\gamma}E_{\alpha\lambda;\gamma}+\eta^{\epsilon}{}_{\beta\mu\nu}V^{\beta}H^{\nu\lambda}\sigma^{\mu}{}_{\lambda}+3H^{\epsilon\nu}\omega_{\nu}=\frac{1}{3}h^{\epsilon\alpha}\rho_{,\alpha}$$
$$+\frac{\theta}{3}q^{\epsilon}-\frac{1}{2}(\sigma^{\epsilon}{}_{\nu}-3\omega^{\epsilon}{}_{\nu})q^{\nu}+\frac{1}{2}\pi^{\epsilon\mu}a_{\mu}+\frac{1}{2}h^{\epsilon\alpha}\pi_{\alpha}{}^{\nu}{}_{;\nu};\tag{38}$$

$$h^{\epsilon\alpha}h^{\lambda\gamma}H_{\alpha\lambda;\gamma}-\eta^{\epsilon}{}_{\beta\mu\nu}V^{\beta}E^{\nu\lambda}\sigma^{\mu}{}_{\lambda}-3E^{\epsilon\nu}\omega_{\nu}=(\rho+p)\omega^{\epsilon}$$
$$-\frac{1}{2}\eta^{\epsilon\alpha\beta\lambda}V_{\lambda}q_{\alpha;\beta}+\frac{1}{2}\eta^{\epsilon\alpha\beta\lambda}(\sigma_{\mu\beta}+\omega_{\mu\beta})\pi^{\mu}{}_{\alpha}V_{\lambda};\tag{39}$$

$$h_{\mu}{}^{\epsilon}h_{\nu}{}^{\lambda}\dot{H}^{\mu\nu}+\theta H^{\epsilon\lambda}-\frac{1}{2}H_{\nu}{}^{(\epsilon}h_{\mu}^{\lambda)}V^{\mu;\nu}-a_{\alpha}E_{\beta}{}^{(\lambda}\eta^{\epsilon)\gamma\alpha\beta}V_{\gamma}$$
$$+\eta^{\lambda\nu\mu\gamma}\eta^{\epsilon\beta\tau\alpha}V_{\mu}V_{\tau}H_{\alpha\gamma}\theta_{\nu\beta}+\frac{1}{2}E_{\beta}{}^{\mu}{}_{;\alpha}h_{\mu}^{(\epsilon}\eta^{\lambda)\gamma\alpha\beta}V_{\gamma}=$$
$$-\frac{3}{4}q^{(\epsilon}\omega^{\lambda)}+\frac{1}{2}h^{\epsilon\lambda}q^{\mu}\omega_{\mu}+\frac{1}{4}\sigma_{\beta}{}^{(\epsilon}\eta^{\lambda)\alpha\beta\mu}V_{\mu}q_{\alpha}+$$
$$+\frac{1}{4}h^{\nu(\epsilon}\eta^{\lambda)\alpha\beta\mu}V_{\mu}\pi_{\nu\alpha;\beta};\tag{40}$$

$$h_{\mu}{}^{\epsilon}h_{\nu}{}^{\lambda}\dot{E}^{\mu\nu}+\theta E^{\epsilon\lambda}-\frac{1}{2}E_{\nu}{}^{(\epsilon}h^{\lambda)}{}_{\mu}V^{\mu;\nu}+a_{\alpha}H_{\beta}{}^{(\lambda}\eta^{\epsilon)\gamma\alpha\beta}V_{\gamma}+$$
$$+\eta^{\lambda\nu\mu\gamma}\eta^{\epsilon\beta\tau\alpha}V_{\mu}V_{\tau}E_{\alpha\gamma}\theta_{\nu\beta}-\frac{1}{2}H_{\beta}{}^{\mu}{}_{;\alpha}h_{\mu}^{(\epsilon}\eta^{\lambda)\gamma\alpha\beta}V_{\gamma}=$$
$$\frac{1}{6}h^{\epsilon\lambda}(q^{\mu}{}_{;\mu}-q^{\mu}a_{\mu}-\pi^{\mu\nu}\sigma_{\mu\nu})-\frac{1}{2}(\rho+p)\sigma^{\epsilon\lambda}+\frac{1}{2}q^{(\epsilon}a^{\lambda)}+$$
$$-\frac{1}{4}h^{\mu(\epsilon}h^{\lambda)\alpha}q_{\mu;\alpha}+\frac{1}{2}h_{\alpha}{}^{\epsilon}h_{\mu}{}^{\lambda}\dot{\pi}^{\alpha\mu}+\frac{1}{4}\pi_{\beta}{}^{(\epsilon}\sigma^{\lambda)\beta}+$$
$$-\frac{1}{4}\pi_{\beta}{}^{(\epsilon}\omega^{\lambda)\beta}+\frac{1}{6}\theta\pi^{\epsilon\lambda}.\tag{41}$$

It is easy to see the similitude to Maxwell's theory. The conservation law of the energy-momentum tensor $T^{\mu\nu}{}_{;\nu}=0$ gives

$$\dot{\rho}+(\rho+p)\theta+\dot{q}^{\mu}V_{\mu}+q^{\alpha}{}_{;\alpha}-\pi^{\mu\nu}\sigma_{\mu\nu}=0,\tag{42}$$

$$(\rho+p)a_{\alpha}-p_{\mu}h^{\mu}{}_{\alpha}+\dot{q}_{\mu}h^{\mu}{}_{\alpha}+\theta q_{\alpha}+q^{\nu}\theta_{\alpha\nu}+$$
$$+q^{\nu}\omega_{\alpha\nu}+\pi_{\alpha}{}^{\nu}{}_{;\nu}+\pi^{\mu\nu}\sigma_{\mu\nu}V_{\alpha}=0.\tag{43}$$

The integrability condition can be translated into equations for the kinematical quantities. Thereby, these evolution equations are

$$\dot{\theta}+\frac{\theta^2}{3}+2(\sigma^2+\omega^2)-a^{\alpha}{}_{;\alpha}=-\frac{1}{2}(\rho+3p),\tag{44}$$

$$h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}\dot{\sigma}_{\mu\nu}+\frac{1}{3}h_{\alpha\beta}(a^{\lambda}{}_{;\lambda}-2\sigma^2-2\omega^2)+a_{\alpha}a_{\beta}+$$
$$-\frac{1}{2}h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}(a_{\mu;\nu}+a_{\nu;\mu})+\frac{2}{3}\theta\sigma_{\alpha\beta}+\sigma_{\alpha\mu}\sigma^{\mu}{}_{\beta}+\tag{45}$$
$$+\omega_{\alpha\mu}\omega^{\mu}{}_{\beta}=-E_{\alpha\beta}-\frac{1}{2}\pi_{\alpha\beta},$$

$$h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}\dot{\omega}_{\mu\nu}-\frac{1}{2}h_{\alpha}{}^{\mu}h_{\beta}{}^{\nu}(a_{\mu;\nu}-a_{\nu;\mu})+\frac{2}{3}\theta\omega_{\alpha\beta}+\tag{46}$$
$$-\sigma_{\beta\mu}\omega^{\mu}{}_{\alpha}+\sigma_{\alpha\mu}\omega^{\mu}{}_{\beta}=0,$$

together with the constraint equations

$$\frac{2}{3}\theta_{,\mu}h^{\mu}{}_{\lambda}-(\sigma^{\alpha}{}_{\gamma}+\omega^{\alpha}{}_{\gamma})_{;\alpha}h^{\gamma}{}_{\lambda}-a^{\nu}(\sigma_{\lambda\nu}+\omega_{\lambda\nu})=-q_{\lambda},\tag{47}$$

$$\omega^{\alpha}{}_{;\alpha}+2\omega^{\alpha}a_{\alpha}=0,\tag{48}$$

$$H_{\tau\lambda}=-\frac{1}{2}h_{(\tau}{}^{\epsilon}h_{\lambda)}{}^{\alpha}\eta_{\epsilon}{}^{\beta\gamma\nu}V_{\nu}(\sigma_{\alpha\beta}+\omega_{\alpha\beta})_{;\gamma}+a_{(\tau}\omega_{\lambda)}.\tag{49}$$

This set of equations corresponds to the necessary equations to propagate GR on a given hypersurface to the whole spacetime.

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